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LETTER TO THE EDITOR

A completely integrable system for the SU(3) Yang–Mills equations

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Abstract. The eight-dimensional Riemannian manifold which describes, via the theory of harmonic maps, the self-dual SU(3) Yang–Mills equations is completely integrable. Sixteen independent Killing fields are determined, two groups of transformations which map solutions into solutions are constructed, and a conjecture on the complete integrability of systems by the inverse scattering method is formulated.

In a previous paper (Xanthopoulos 1981, to be referred to as I) we described the SU(3) self-dual Yang–Mills equations in terms of a suitable harmonic map of Riemannian manifolds. Here are some of the conclusions of I.

(i) The information of the self-dual, SU(3) Yang–Mills equations written in the R gauge, the equations (I4), is coded in the structure of the Riemannian manifold (N, g'_{ab}) with line element, in the coordinate chart $\{f^a\} = (u, v, x_1, x_2, y_1, y_2, \omega_1, \omega_2)$, given by

$$\mathrm{d}s^2 = g'_{ab}(\mathrm{d}f^a)(\mathrm{d}f^b) = \frac{1}{u^2}(\mathrm{d}u)^2 + \frac{1}{v^2}(\mathrm{d}v)^2 - \frac{1}{uv}(\mathrm{d}u)(\mathrm{d}v) + \frac{u}{v^2}(\mathrm{d}y_1)(\mathrm{d}y_2)$$
$$+ \left(\frac{y_1 y_2}{uv} + \frac{v}{u^2}\right)(\mathrm{d}x_1)(\mathrm{d}x_2) - \frac{y_1}{uv}(\mathrm{d}x_1)(\mathrm{d}\omega_2) - \frac{y_2}{uv}(\mathrm{d}\omega_1)(\mathrm{d}\omega_2) + \frac{1}{uv}(\mathrm{d}\omega_1)(\mathrm{d}\omega_2). \tag{1}$$

(ii) The Killing (vector) fields (Eisenhart 1926) of (N, g'_{ab}) provide infinitesimal transformations, and the isometries of (N, g'_{ab}) finite transformations, which generate from the solutions of equations (I4) more general solutions.

(iii) The geodesics of (N, g'_{ab}) provide certain functionally dependent solutions of the equations (I4). (Throughout the paper geodesics are always meant to be affinely parametrised.) Note that (ii) and (iii) are related: Killing fields lead to constants of the geodesic motions which are useful for the explicit determination of the geodesics.

The purpose of this Letter is twofold: first, to report the explicit determination of sixteen linearly independent Killing fields of the Riemannian manifold (N, g'_{ab}), and second, to formulate a conjecture, and to present the evidence which supports its correctness, for the complete integrability of certain systems of partial differential equations.

The sixteen linearly independent Killing fields of (N, g'_{ab}) are:

$$A_1^a = (0, 0, 1, 0, 0, 0, 0, 0)$$
$$A_2^a = (0, 0, 0, 1, 0, 0, 0, 0)$$
$$B_1^a = 0, 0, 0, 0, 0, 0, 1, 0)$$

$$B_2^a = (0, 0, 0, 0, 0, 0, 0, 1)$$
$$K_1^a = (-ux_2, 0, u^2v^{-1}, -x_2^2, 0, x_2y_2-\omega_2, u^2v^{-1}y_1, -x_2\omega_2)$$
$$K_2^a = (-ux_1, 0, -x_1^2, u^2v^{-1}, x_1y_1-\omega_1, 0, -x_1\omega_1, u^2v^{-1}y_2)$$
$$\Lambda_1^a = (-u\omega_2, v(x_2y_2-\omega_2), u^2v^{-1}y_2, -x_2\omega_2, -v^2u^{-1}x_2, y_2(x_2y_2-\omega_2), uv+u^2v^{-1}y_1y_2, -\omega_2^2)$$
$$\Lambda_2^a = (-u\omega_1, v(x_1y_1-\omega_1), -x_1\omega_1, u^2v^{-1}y_1, y_1(x_1y_1-\omega_1), -v^2u^{-1}x_1, -\omega_1^2, uv+u^2v^{-1}y_1y_2)$$
$$E_1^a = (2u/3, v/3, 0, x_2, 0, 0, 0, \omega_2)$$
$$E_2^a = (2u/3, v/3, x_1, 0, 0, 0, \omega_1, 0)$$
$$H_1^a = (u/3, 2v/3, 0, 0, 0, y_2, 0, \omega_2)$$
$$H_2^a = (u/3, 2v/3, 0, 0, y_1, 0, \omega_1, 0)$$
$$P_1^a = (0, 0, 0, 0, 0, 1, 0, x_2)$$
$$P_2^a = (0, 0, 0, 0, 1, 0, x_1, 0)$$
$$Q_1^a = (0, -vy_2, 0, \omega_2, v^2u^{-1}, -y_2^2, 0, 0)$$
$$Q_2^a = (0, -vy_1, \omega_1, 0, -y_1^2, v^2u^{-1}, 0, 0). \tag{2}$$

In the above list, $A_i^a = (\partial/\partial x_i)^a$ and $B_i^a = (\partial/\partial\omega_i)^a$, $i = 1, 2$, are the four obvious Killing fields, associated with the four ignorable coordinates x_i and ω_i, $i = 1, 2$. The existence of the remaining twelve Killing fields is not apparent. In fact, even the straightforward verification that they are Killing fields—i.e. the verification that they satisfy Killing's equation $2\nabla_{(a}K_{b)} = \nabla_aK_b + \nabla_bK_a = 0$ with ∇_a the derivative operator compatible with g'_{ab}—requires a great deal of algebraic computation. They were found by considering the geodesic equations of g'_{ab} and writing some of them as vanishing total derivatives with respect to the affine parameter s. It is clear that if $\mathrm{d}(\xi^bK_b)/\mathrm{d}s = \xi^a\nabla_a(\xi^bK_b) = \xi^a\xi^b\nabla_aK_b = 0$ for the arbitrary geodesic vector field† $\xi^a = (\dot u, \dot v, \dot x_1, \dot x_2, \dot y_1, \dot y_2, \dot\omega_1, \dot\omega_2)$, then $\nabla_{(a}K_{b)} = 0$, which implies that K_a is a Killing field. Subsequently, additional Killing fields were obtained by commuting the already known ones, a step in which it was used that the commutator $[K^a, \Lambda^a] = K^m\nabla_m\Lambda^a - \Lambda^m\nabla_mK^a$ of any two Killing fields is also a Killing field. We should mention at this point that we do not have a proof that we have found all the Killing fields of the metric g'_{ab}. However, from the commutation relations (3) it follows that the Killing fields (2) form a sixteen-dimensional Lie algebra L. The structure of L is determined by the following commutation relations:

$$\begin{array}{lll}
[A_\alpha, E_\beta] = A_\alpha & [A_\alpha, K_\beta] = H_\beta - 2E_\beta & [A_\alpha, P_\beta] = B_\alpha \\
[B_\alpha, E_\beta] = B_\alpha & [A_\alpha, \Lambda_\beta] = -Q_\beta & [B_\alpha, Q_\beta] = A_\alpha \\
[B_\alpha, H_\beta] = B_\alpha & [B_\alpha, K_\beta] = -P_\beta & [P_\alpha, Q_\alpha] = E_\alpha - 2H_\alpha \\
[K_\alpha, E_\alpha] = K_\alpha & [B_\alpha, \Lambda_\beta] = -(E_\beta + H_\beta) & [P_\alpha, H_\alpha] = P_\alpha \\
[\Lambda_\alpha, E_\alpha] = -\Lambda_\alpha & [K_\alpha, Q_\alpha] = -\Lambda_\alpha & [Q_\alpha, H_\alpha] = -Q_\alpha \\
[\Lambda_\alpha, H_\alpha] = -\Lambda_\alpha & [\Lambda_\alpha, P_\alpha] = -K_\alpha. &
\end{array} \tag{3}$$

† The 'dot' denotes differentiation with respect to the affine parameter s.

Here, and in the sequel, the indices α and β take the values 1 and 2, $\alpha \neq \beta$ when they both appear in the same expression, and repeated α's or β's do not imply summation. All the commutators of L which are not listed in (3) are equal to zero. Obviously, the algebra L possesses three four-dimensional Abelian subalgebras, the subalgebras generated by $\{A_\alpha, B_\alpha, \alpha = 1, 2\}$, $\{K_\alpha, \Lambda_\alpha, \alpha = 1, 2\}$ and $\{E_\alpha, H_\alpha, \alpha = 1, 2\}$.

The Killing fields lead immediately to the infinitesimal transformation

$$f^a \to \tilde{f}^a = f^a + \varepsilon \boldsymbol{K}^a(f^b) + \mathrm{O}(\varepsilon^2) \tag{4}$$

which, to first order in the parameter ε, maps solutions into solutions of equation (I4), where K^a are now the contravariant components of any (constant) linear combination of the Killing fields. Moreover, we were able to exponentiate explicitly (cf I, § 5) some of the isometries generated by the Killing fields (2) and obtain exact transformations which map solutions into solutions. From the Killing field $\alpha_1 K_2 + \alpha_2 K_1$ we obtained the transformation

$$\begin{aligned}
&\tilde{u} = \Omega^{-1} u \qquad \tilde{v} = v \\
&\tilde{x}_1 = \Omega^{-1}[x_1 + \alpha_2(x_1 x_2 + u^2 v^{-1})] \qquad \tilde{y}_1 = y_1 + \alpha_1(x_1 y_1 - \omega_1) \\
&\tilde{\omega}_1 = \Omega^{-1}[\omega_1 + \alpha_2(x_2\omega_1 + y_1 u^2 v^{-1})] \\
&\Omega = 1 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_1\alpha_2(x_1 x_2 + u^2 v^{-1})
\end{aligned} \tag{5}$$

and from the Killing field $\alpha_1\Lambda_2 + \alpha_2\Lambda_1$ the transformation

$$\begin{aligned}
&\tilde{u} = P^{-1} u \qquad \tilde{v} = Q^{-1} v \\
&\tilde{x}_1 = x_1(\rho_1 - 1)(\rho_1 P)^{-1} \qquad \tilde{\omega}_1 = P^{-1}[\omega_1 + \alpha_2(\omega_1\omega_2 + y_1 y_2 u^2 v^{-1} + uv)] \\
&\tilde{y}_1 = \frac{\rho_1}{x_1(\rho_1 - 1)Q}\{x_1 y_1 + \alpha_2[x_1 y_1(2\omega_2 - x_2 y_2) + u^2 y_1 y_2 v^{-1} - v^2 x_1 x_2 u^{-1}] \\
&\qquad + \alpha_2^2[\omega_2(x_1 y_1 \omega_2 + y_1 y_2 u^2 v^{-1} - x_1 x_2 v^2 u^{-1}) \\
&\qquad - x_2 y_2(x_1 y_1 \omega_2 + y_1 y_2 u^2 v^{-1} + uv)]\} \\
&P = 1 + \alpha_1\omega_1 + \alpha_2\omega_2 + \alpha_1\alpha_2(\omega_1\omega_2 + y_1 y_2 u^2 v^{-1} + uv) \\
&Q = P - \alpha_1 x_1 y_1(1 + \alpha_2\omega_2) - \alpha_2 x_2 y_2(1 + \alpha_1\omega_1) \\
&\qquad + \alpha_1\alpha_2(x_1 x_2 v^2 u^{-1} - y_1 y_2 u^2 v^{-1} + x_1 x_2 y_1 y_2) \\
&A = \alpha_1\alpha_2 u(y_1 y_2 u^2 v^{-1} + x_1 x_2 \omega_1\omega_2 v u^{-2} + x_1 y_1 \omega_2 + x_2 y_2 \omega_1) \\
&B = (\alpha_1\omega_1 + \alpha_2\omega_2)x_1 x_2 v u^{-1} + u(\alpha_1 x_1 y_1 + \alpha_2 x_2 y_2) \qquad \Gamma = x_1 x_2 v u^{-1}.
\end{aligned} \tag{6}$$

ρ_1 and ρ_2 are the two roots of the equation $A\rho^2 + B\rho + \Gamma = 0$, where α_1 and α_2 are arbitrary constants. In equations (5) and (6) the non-listed variables $\tilde{x}_2$, $\tilde{y}_2$, $\tilde{\omega}_2$ are obtained from the expressions for $\tilde{x}_1$, $\tilde{y}_1$, $\tilde{\omega}_1$ respectively by interchanging 1 and 2. The transformations preserve the reality of the corresponding Yang–Mills fields when the constants α_1 and α_2 are complex conjugate. We should point out, however, that we do not know *a priori* whether these transformations lead to physically distinct solutions or to solutions which are related to the original ones by a gauge transformation.

By using the Killing fields (2) it is straightforward to obtain the following sixteen integrals of the geodesic motions:

$$2a_\alpha = vu^{-2}\dot{x}_\beta - y_\alpha(uv)^{-1}(\dot{\omega}_\beta - y_\beta\dot{x}_\beta) \qquad 2b_\alpha = (uv)^{-1}(\dot{\omega}_\beta - y_\beta\dot{x}_\beta)$$

$$\begin{aligned}
2k_\alpha &= -2x_\beta u^{-1}\dot{u} + x_\beta v^{-1}\dot{v} - vu^{-2}x_\beta^2\dot{x}_\alpha + uv^{-2}(x_\beta y_\beta - \omega_\beta)\dot{y}_\alpha + \dot{x}_\beta \\
&\quad + x_\beta(x_\beta y_\beta - \omega_\beta)(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha) \\
2\lambda_\alpha &= -(x_\beta y_\beta + \omega_\beta)u^{-1}\dot{u} + (2x_\beta y_\beta - \omega_\beta)v^{-1}\dot{v} - vu^{-2}x_\beta\omega_\beta\dot{x}_\alpha \\
&\quad + (x_\beta y_\beta - \omega_\beta)\omega_\beta(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha) + uv^{-2}y_\beta(x_\beta y_\beta - \omega_\beta)\dot{y}_\alpha + \dot{\omega}_\beta - x_\beta\dot{y}_\beta
\end{aligned} \tag{7}$$

$$2\varepsilon_\alpha = u^{-1}\dot{u} + vu^{-2}x_\beta\dot{x}_\alpha + (\omega_\beta - x_\beta y_\beta)(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha)$$

$$2\eta_\alpha = v^{-1}\dot{v} + uv^{-2}y_\beta\dot{y}_\alpha + \omega_\beta(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha)$$

$$2p_\alpha = uv^{-2}\dot{y}_\alpha + x_\beta(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha)$$

$$2q_\alpha = y_\beta u^{-1}\dot{u} - 2y_\beta v^{-1}\dot{v} + \omega_\beta vu^{-2}\dot{x}_\alpha - uv^{-2}y_\beta^2\dot{y}_\alpha + \dot{y}_\beta - y_\beta\omega_\beta(uv)^{-1}(\dot{\omega}_\alpha - y_\alpha\dot{x}_\alpha)$$

where we have set $a_\alpha = A_{\alpha,m}\xi^m$, $b_\alpha = B_{\alpha,m}\xi^m$, $K_\alpha = K_{\alpha,m}\xi^m$, $\lambda_\alpha = \Lambda_{\alpha,m}\xi^m$, $\varepsilon_\alpha = E_{\alpha,m}\xi^m$, $\eta_\alpha = H_{\alpha,m}\xi^m$, $p_\alpha = P_{\alpha,m}\xi^m$ and $q_\alpha = Q_{\alpha,m}\xi^m$. Since the integrals do not depend explicitly on the affine parameter of the geodesics, all sixteen of them cannot be functionally independent. We have verified, however, that any fifteen of them are indeed independent and therefore we conclude that the Riemannian manifold (N, g'_{ab}) is completely integrable. To clarify the terminology, we mention that a manifold with metric is called 'completely integrable' when its geodesics can be explicitly determined completely algebraically, while it is called simply 'integrable' when the integration of its geodesics can be reduced to mere quadratures (equivalently, when it possesses $2n-1$ and n respectively global integrals of motion, where n is the dimension of the manifold). Finally we note that from the integrals (7) we can immediately write corresponding integrals for the SU(3) Yang–Mills equations (14).

Next we turn to the formulation of the conjecture. In applications of the inverse scattering method the terminology 'completely integrable' is used differently from the way in which it is used in differential geometry. In the inverse scattering case a system of partial differential equations is called completely integrable when it admits a linear eigenvalue problem (in the spirit of Lax), (Lax 1968, Scott *et al* 1973) with compatibility conditions the differential equations of the system. Motivated from the use of the same terminology, we were led to formulate the following.

Conjecture. For those systems of differential equations which can be described both as a linear eigenvalue problem and as a harmonic map of Riemannian manifolds, the two notions of integrability are equivalent.

In addition to the justification it will provide for the use of the same terminology, the validity of the conjecture will lead to a nice geometrical characterisation of a certain class of systems of partial differential equations for which the inverse scattering method is applicable: the system is integrable in the sense of the inverse scattering method if and only if the corresponding Riemannian manifold of the harmonic map description of the system admits $2n-1$ independent Killing vector fields (more generally, $2n-1$ independent, irreducible Killing tensor fields (Sommers 1973)).

Finally, we mention a few examples for which the validity of the conjecture has been established.

(a) The stationary axisymmetric Einstein vacuum equations (sometimes referred also as the Ernst (1968a) equation). Linear eigenvalue problems have been formulated by Maison (1978, 1979), Harrison (1978) and Belinskii and Zakharov (1978, 1979) while the last authors have also developed a method for solving these equations. The equations can also be described as a harmonic map (Matzner and Misner 1967) in which the manifold of fields is the two-dimensional hyperboloid of constant scalar curvature with line element $ds^2 = x^{-2}[(dx)^2 + (dy)^2]$. This manifold admits three independent Killing fields and therefore it is completely integrable in both uses of the terminology.

(b) The stationary axisymmetric Einstein–Maxwell electrovacuum equations (Ernst 1968b). A linear eigenvalue problem has been formulated and some progress towards its solution has been made by Aleksejev (1980). As a harmonic map the equations can be described (Hoenselaers 1978, Kramer *et al* 1980) as a harmonic map with manifold of fields a four-dimensional manifold admitting eight linearly independent Killing vector fields, the generators of the eight-dimensional group of the Ehlers–Harrison–Kinnersley gauge transformations (Kinnersley 1973).

(c) The SU(2) and SU(3) self-dual Yang–Mills source-free equations. Their integrability by the inverse scattering method in $2+1$ dimensions has been shown by Manakov and Zakharov (1981). On the other hand, the auxiliary (cf I) systems of these equations can be described as harmonic maps with manifolds of fields a three- (Nutku 1978) and an eight- (Xanthopoulos 1981) dimensional manifold, respectively. The three-dimensional manifold is a hyperboloid of constant scalar curvature with line element $ds^2 = x^{-2}[(dx)^2 + (dy)(dz)]$; it is completely integrable because it admits $3(3+1)/2 = 6$ independent Killing fields. The eight-dimensional manifold is the manifold with line element (1); its complete integrability has been established by the results of the present paper.

(d) The two-dimensional nonlinear σ models. Their complete integrability has been demonstrated by the general theory of Zakharov and Mikhailov (1978). In their harmonic map descriptions the manifolds of fields are unit hyperboloids which are completely integrable because they admit the maximum number of independent Killing fields allowed by their dimensionality.

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